SAMPLE

MATH 5A - FINAL EXAM
Fall 2023

200 POINTS
Instructions on Canvas.

(1) Find the following limits if they exist (if the limit is ∞ or $-\infty$ say so). (12 points)

$(a)\ \lim\limits_{x\to 9}\dfrac{9-x}{\sqrt{x}-3}$ ___-6___ $(b)\ \lim\limits_{x\to 5^+}\dfrac{-x^2}{x-5}$ ___$-\infty$___

$$\frac{9-x}{\sqrt{x}-3}\cdot\frac{\sqrt{x}+3}{\sqrt{x}+3}=\frac{\cancel{(9-x)}^{-1}(\sqrt{x}+3)}{\cancel{x-9}}=-(\sqrt{x}+3)$$

"$\dfrac{-25}{+0}$"

$\dfrac{\text{non zero}}{0}$ $\to \infty$, $\to -\infty$, $\to$ DNE

(2) If $f(x)=\begin{cases}5+x \text{ if } x>2\\ 15-x^3 \text{ if } x\le 2\end{cases}$ (6 points)

(a) Find

$\lim\limits_{x\to 2^-} f(x)$ ___7___

$\lim\limits_{x\to 2^+} f(x)$ ___7___

$\lim\limits_{x\to 2} f(x)$ ___7___

(b) Is $f(x)$ continuous for all real numbers? ___yes___

(3) Given $f(x)=2x^3-9x^2+12x-3$ (14 points)

(a) Find coordinates of local extrema (if any)
(b) Find absolute extrema on $[0,3]$

Critical #s

$f'(x)=0$
$6x^2-18x+12=0$
$6(x^2-3x+2)=0$
$6(x-2)(x-1)=0$
$x=2,1$

$\Rightarrow$ f' sign chart: $0 \quad + \quad 1 \quad - \quad 2 \quad + \quad 3$

x	0	1	2	3
$f(x)$	-3	2	1	6

Coordinates of Local Maximum(s): ___(1, 2)___

Coordinates of Local Minimum(s): ___(2, 1)___

Absolute Max Value: ___6___

Absolute Min Value: ___-3___

on $[0,3]$

(4) Find the derivative, $\dfrac{dy}{dx}$ and <u>simplify your answer</u>: (30 points)

(a) $y = \dfrac{x^5}{x^3 + 2x}$

$$\frac{dy}{dx} = \frac{(x^3+2x)5x^4 - x^5(3x^2+2)}{(x^3+2x)^2} = \frac{5x^7+10x^5-3x^7+2x^5}{(x^3+2x)^2}$$

$$= \frac{2x^7+12x^5}{(x^3+2x)^2} = \frac{2x^5(x^2+6)}{x^2(x^2+2)^2} = \frac{2x^3(x^2+6)}{(x^2+2)^2}$$

(b) $y = \tan\left(\cos\left(x^2\right)\right)$

$$\frac{dy}{dx} = \sec^2(\cos(x^2))\,\frac{d}{dx}(\cos(x^2))$$

$$= \sec^2(\cos(x^2)) \cdot -\sin(x^2)\,\frac{d}{dx}(x^2)$$

$$= -2x\sec^2(\cos(x^2))\sin(x^2)$$

(c) $y = \dfrac{3x}{\sqrt{x^2-4}} = 3x(x^2-4)^{-1/2}$

$$\frac{dy}{dx} = 3(x^2-4)^{-1/2} - \frac{3}{2}x(x^2-4)^{-3/2}(2x)$$

$$= 3(x^2-4)^{-3/2}(x^2-4-x^2)$$

$$= \frac{-12}{(x^2-4)^{3/2}}$$

(5) Integrate: (60 points)

(a) $\int 4x\sqrt{x^2-5}\,dx$

$u = x^2-5$

$du = 2x\,dx$

$4 \cdot \frac{1}{2}\int u^{1/2}\,du$

$= 2 \quad \frac{2}{3}u^{3/2} + C$

$= \frac{4}{3}(x^2-5)^{3/2} + C$

Remember, you can check by differentiating

(b) $\int \frac{1}{\cos^2 x\sqrt{1+\tan x}}\,dx$

$u = 1+\tan x$

$du = \sec^2 x\,dx$

$= \int \frac{\sec^2 x}{\sqrt{1+\tan x}}\,dx$

$= \int \frac{1}{\sqrt{u}}\,du$

$= \int u^{-1/2}\,du$

$= 2u^{1/2} + C$

$= 2\sqrt{1+\tan x} + C$

(c) $\int_0^{\pi/3} \cos^4 x \sin x\,dx$

$u = \cos x$

$du = -\sin x\,dx$

$-\int_1^{1/2} u^4\,du$

$\int_{1/2}^{1} u^4\,du$

$\frac{32}{5}$

$\frac{1}{5}u^5\Big]_{1/2}^{1}$

$= \frac{1}{5}\left(1-\frac{1}{32}\right) = \frac{31}{160}$

(d) $\int \sec^2\left(\frac{x}{4}\right)dx$

$u = \frac{x}{4}$

$du = \frac{1}{4}dx$

$4\int \sec^2 u\,du$

$= 4\tan u + C$

$= 4\tan\left(\frac{x}{4}\right) + C$

(5 continued)

(d) $\int x^4\left(x^2 - x^{3/2}\right)dx$

$$\int (x^6 - x^{\frac{11}{2}})\,dx$$

$$\boxed{\frac{1}{7}x^7 + \frac{2}{13}x^{\frac{13}{2}} + c}$$

(e) $\int_{-2}^{2} |x-1|\,dx$

$$|x-1| = \begin{cases} x-1 & \text{if } x-1 \geq 0 \Rightarrow x \geq 1 \\ -(x-1) & \text{if } x-1<0 \Rightarrow x<1 \end{cases}$$

$$\int_{-2}^{1} -(x-1)\,dx + \int_{1}^{2} (x-1)\,dx$$

$$-\frac{x^2}{2} + x\Big]_{-2}^{1} + \frac{1}{2}x^2 - x\Big]_{1}^{2}$$

$$\frac{1}{2} - (-4) + 0 - \left(-\frac{1}{2}\right)$$

$$\frac{9}{2} + \frac{1}{2} = \boxed{5}$$

(6) Air is being pumped into a spherical balloon at a rate of 5 cm³/min. Determine the rate at which the radius of the balloon is increasing when the diameter of the balloon is 3 cm. (give units)

(Volume of a sphere is $\frac{4}{3}\pi r^3$)

(12 points)

Know

$$\frac{dV}{dt} = 5\,\frac{cm^3}{min}$$

want

$$\left.\frac{dr}{dt}\right|_{r=3/2}$$

relate

$$\frac{d}{dt}\,V = \frac{d}{dt}\,\frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$

when $r = 3/2$

$$\frac{dr}{dt} = \frac{1}{4\pi\left(\frac{3}{2}\right)^2}\cdot 5$$

$$= \boxed{\frac{5}{9\pi}\ cm/min}$$

(7) Consider the shaded region bound by $y=\sqrt{x}$, $y=2-x$, and the x axis as shown. Set up (do not evaluate) integrals for each of the following situations.

(25 points)

a) The area of the shaded region with respect to x

Top curve changes - need 2 integrals.

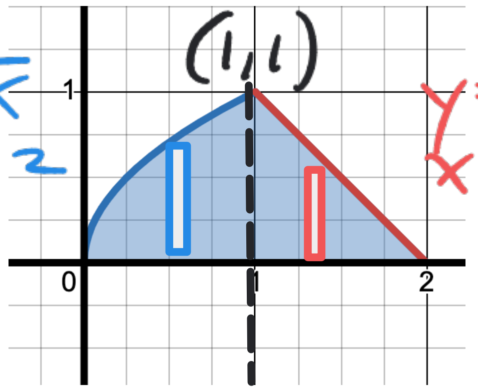
$y=\sqrt{x}$

$x=y^2$

(1,1)

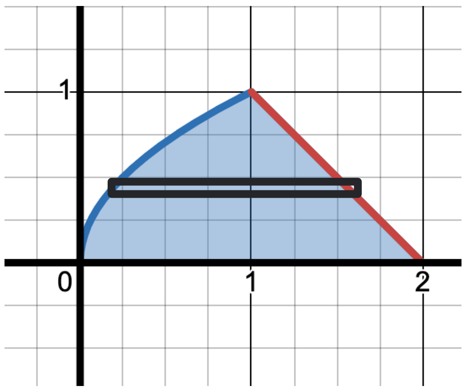
$y=2-x$

$x=2-y$



$$\int_0^1 \sqrt{x}\,dx + \int_1^2 (2-x)\,dx$$

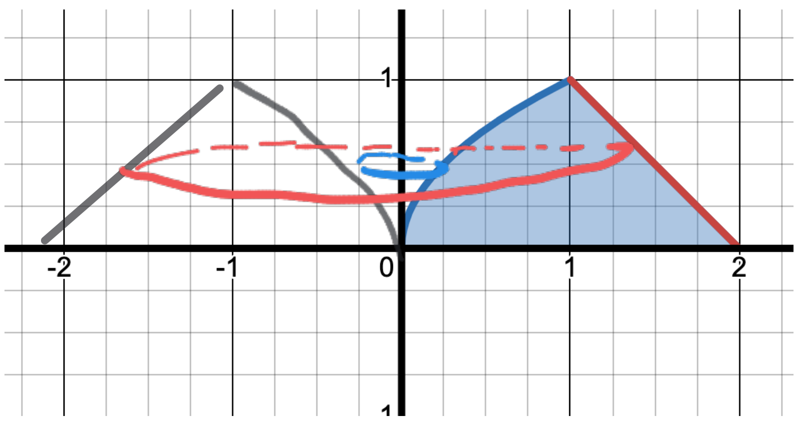
b) The area of the shaded region with respect to y



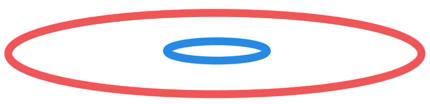
$$\int_0^1 (2-y - y^2)\,dy$$

c) The volume of the solid generated when the region is revolved about the y axis, using disks/washers.

$$\pi(R_o^2 - R_i^2)$$



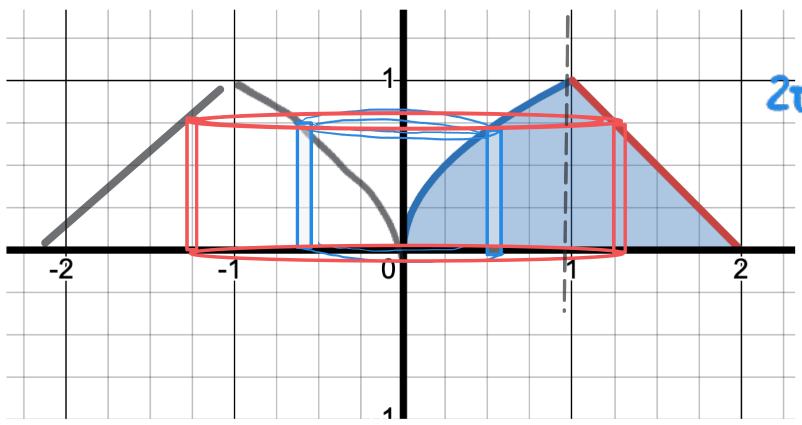
$$\pi \int_0^1 \left((2-y)^2 - (y^2)^2\right) dy$$



d) The volume of the solid generated when the region is revolved about the y axis, using cylindrical shells

Top curve changes

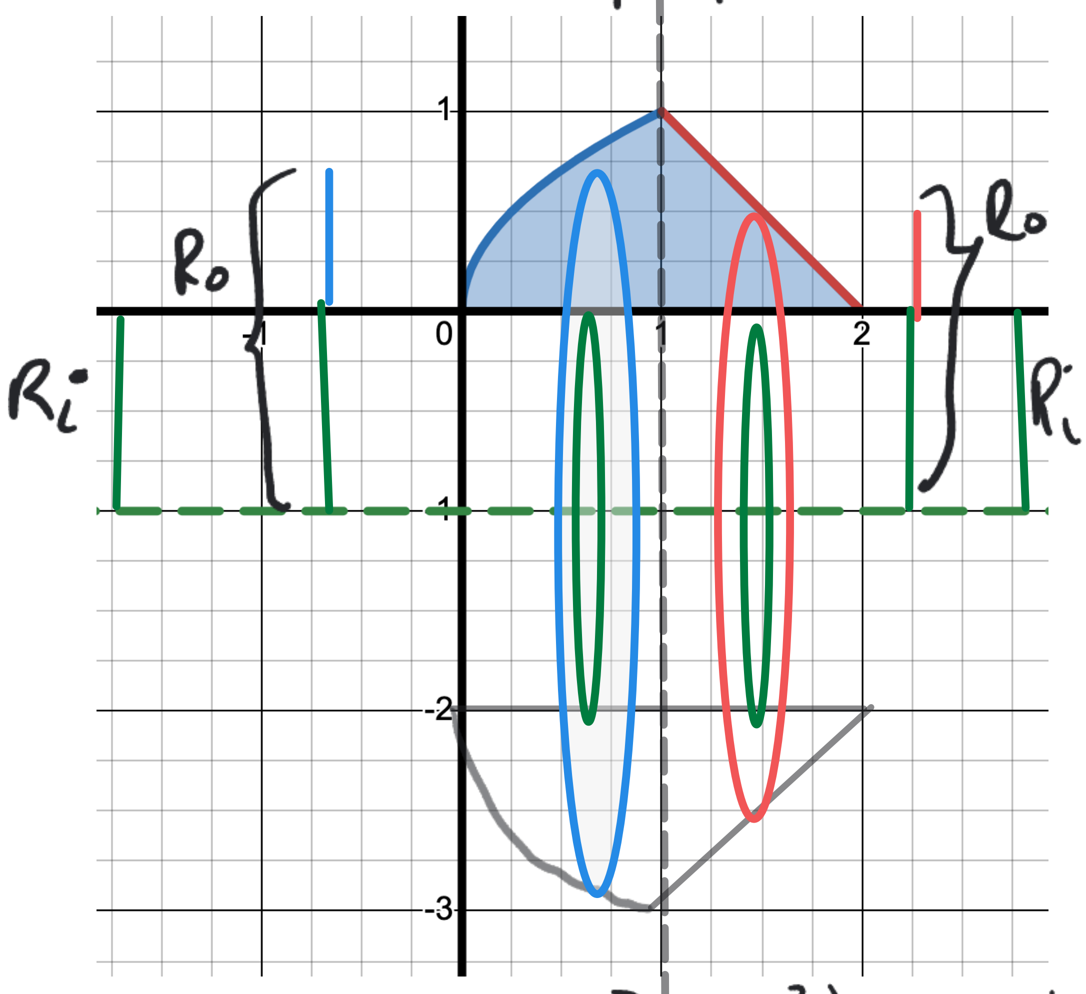
$2\pi r h \cdot \text{thickness}$

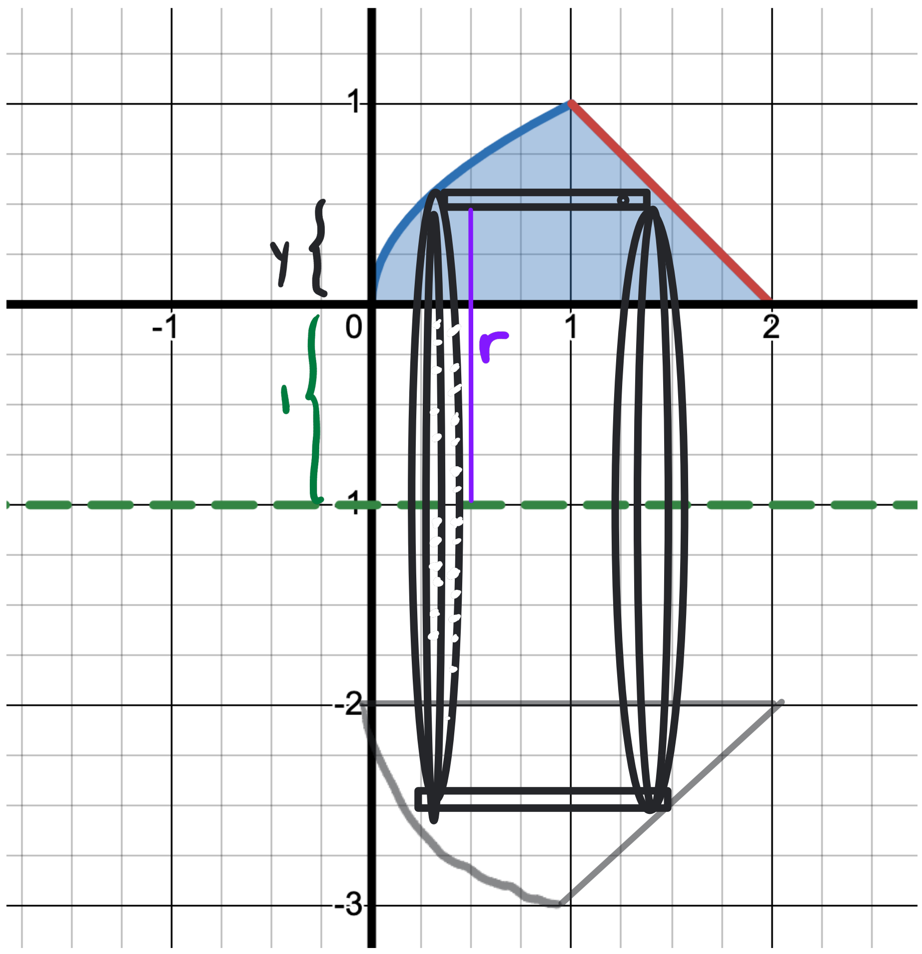


$$2\pi\int_0^1 x\sqrt{x}\,dx + 2\pi\int_1^2 x(2-x)\,dx$$

e) The volume of the solid generated when the region is revolved about the line $y = -1$ using either method.

split





$\pi(R_o^2 - R_i^2)$

$2\pi r h$

$$2\pi\int_0^1 (y+1)(2-y-y^2)\,dy$$

$$\pi\int_0^1 \left((\sqrt{x}- -1)^2 - (1)^2\right)dx$$
$$+\ \pi\int_1^2 \left((2-x- -1)^2 - (1)^2\right)dx$$

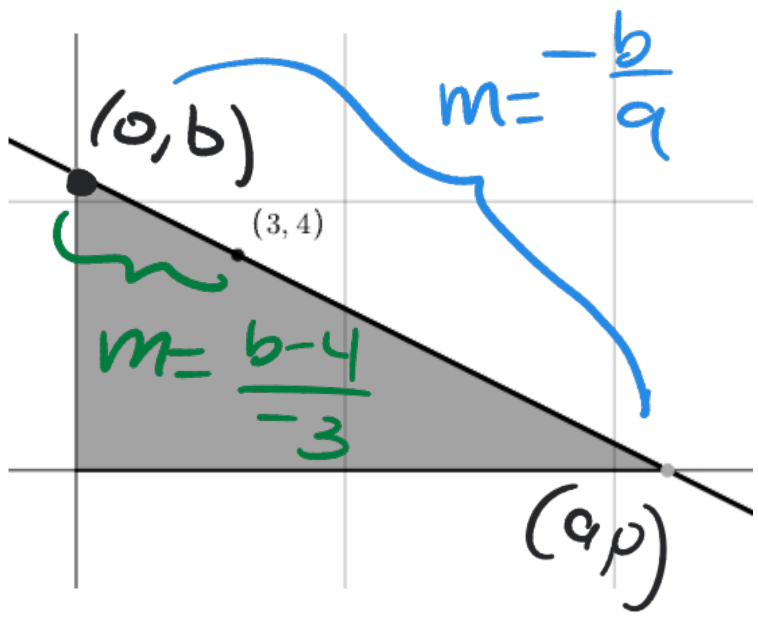
(8) Find the equation of the line through (3,4) which cuts from the first quadrant a triangle of minimum area. Validate how you know you have found the *absolute* min.

(12 points)

Min: $A = \frac{1}{2}ab$

Need to relate a, b

Use slope: $\frac{b-4}{-3} = \frac{-b}{a}$



$$a(b-4) = 3b$$

$$a = \frac{3b}{b-4}$$

$A = \frac{1}{2} \frac{3b}{b-4} b$ domain $b > 4$

$$A = \frac{3}{2} \cdot \frac{b^2}{b-4}.$$

Find crit #s

$$A'(b) = \frac{3}{2} \frac{(b-4)2b - b^2}{(b-4)^2} = \frac{3}{2} \frac{b^2-8b}{(b-4)^2} = \frac{3}{2} \frac{b(b-8)}{(b-4)^2}$$

$A'(b) = 0$ with $b > 4 \Rightarrow b = 8 \Rightarrow a = \frac{3(8)}{8-4} = 6$

$\Rightarrow m = \frac{-b}{a} = \frac{-4}{3}$

Equation of line : $y = \frac{-4}{3}x + 8$

(9) Graph and discuss the function. Show ALL work. Credit will not be given without supporting work and explanation. Be sure to graph x-intercepts, y-intercepts. Clearly show all information found in looking at the first and second derivative (including local extrema, inflection points, vertical and horizontal tangents and end behavior).

$f(x) = x - 4\sqrt{x} = x^{1/2}(x^{1/2} - 4)$ (17 points)

$\boxed{f(x)}$ domain $x \geq 0$

$\lim_{x \to \infty} f(x) = \infty$

X-int $x^{1/2} = 0$ $x^{1/2} = 4$

$x = 0, 16$

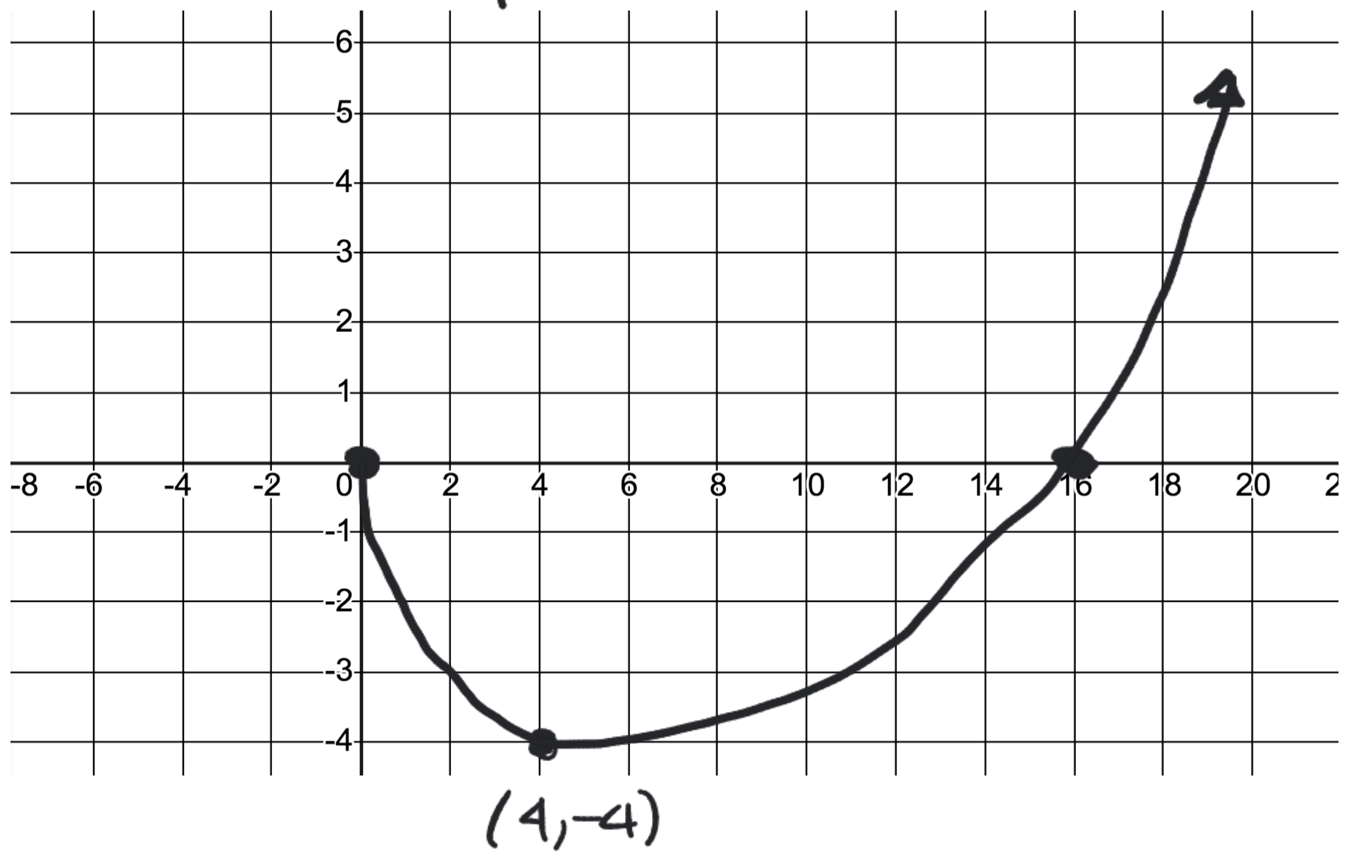
y-int $= 0$

$\boxed{f'(x)}$ $f'(x) = 1 - 2x^{-1/2} = x^{-1/2}(x^{1/2} - 2) = \dfrac{\sqrt{x} - 2}{\sqrt{x}}$

$f'(x) = 0 \Rightarrow \sqrt{x} = 2 \Rightarrow x = 4$

$f'(x)$ undef $\Rightarrow x = 0$

Local min $(4, -4)$



$\boxed{f''(x)}$ $f''(x) = x^{-3/2} > 0$ concave up

(10) Find the equation of the line tangent to the curve $x^2+2xy+4y^2=12$, at the point (2,1).

Slope

Find $\frac{dy}{dx}$ at (2,1)

$$\frac{d}{dx}(x^2+2xy+4y^2)=\frac{d}{dx}(12)$$

$$2x+2y+2x\frac{dy}{dx}+8y\frac{dy}{dx}=0$$

$$(2x+8y)\frac{dy}{dx}=-2x-2y$$

$$\frac{dy}{dx}=\frac{-2(x+y)}{2(x+4y)}$$

$$\frac{dy}{dx}=-\frac{x+y}{x+4y}$$

$$\left.\frac{dy}{dx}\right|_{(2,1)}=-\frac{3}{2+4}=\frac{-3}{6}=\boxed{-\frac{1}{2}}$$

Line $y-1=-\frac{1}{2}(x-2)$

$$\boxed{y=-\frac{1}{2}x+2}$$